$$
\begin{aligned}
[L_m, \alpha_k^\mu] &= \left[ \frac{1}{2} \sum_{n=-\infty}^{\infty} :(\alpha_{m-n} \cdot \alpha_n + \widetilde{\alpha}_{m-n} \cdot \widetilde{\alpha}_n):, \alpha_k^\mu \right] \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} [\alpha_{m-n}, \alpha_k^\mu] \cdot \alpha_n + \alpha_{m-n} \cdot [\alpha_n, \alpha_k^\mu] \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} (m-n)\delta_{m-n+k,0}\alpha_n^\mu + \alpha_{m-n}^\mu n \delta_{n+k,0} \\
&= \frac{1}{2} \Big( (m-(m+k))\alpha_{m+k}^\mu - k\alpha_{m-(-k)}^\mu \Big) \\
&= -k\alpha_{m+k}^\mu
\end{aligned}
$$