

$$\begin{aligned}
\left[ L_0, \frac{1}{2} \sum_{k=-\infty}^{\infty} : \alpha_{n-k} \cdot \alpha_k : \right] &= \frac{1}{2} \sum_{k=-\infty}^{\infty} : [L_0, \alpha_{n-k}] \cdot \alpha_k + \alpha_{n-k} \cdot [L_0, \alpha_k] : \\
&= \frac{1}{2} \sum_{k=-\infty}^{\infty} : (k - n) \alpha_{n-k} \cdot \alpha_k + \alpha_{n-k} \cdot (-k) \alpha_k : \\
&= -n \left( \frac{1}{2} \sum_{k=-\infty}^{\infty} : \alpha_{n-k} \cdot \alpha_k : \right)
\end{aligned}$$