$$
\begin{aligned}
\left[L_m, \frac{1}{2}\sum_{k=-\infty}^{\infty} :\alpha_{n-k}\cdot\alpha_k:\right] &= \frac{1}{2}\sum_{k=-\infty}^{\infty} :[L_m,\alpha_{n-k}]\cdot\alpha_k + \alpha_{n-k}\cdot[L_m,\alpha_k]: \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} :(k-n)\alpha_{m+n-k}\cdot\alpha_k + \alpha_{n-k}\cdot(-k)\alpha_{m+k}: \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} :(k-n)\alpha_{m+n-k}\cdot\alpha_k + \alpha_{n-(k-m)}\cdot(-(k-m))\alpha_{m+(k-m)}: \\
&= (m-n)\frac{1}{2}\sum_{k=-\infty}^{\infty} :\alpha_{m+n-k}\cdot\alpha_k:
\end{aligned}
$$