$$\begin{aligned}
\left[L_1, \sum_{k=0}^{\infty} \alpha_{-1-k} \cdot \alpha_k\right] &= \sum_{k=0}^{\infty}([L_1, \alpha_{-1-k}] \cdot \alpha_k + \alpha_{-1-k} \cdot [L_1, \alpha_k]) \\
&= \sum_{k=0}^{\infty}((k+1)\alpha_{-k} \cdot \alpha_k + \alpha_{-1-k} \cdot (-k)\alpha_{k+1}) \\
&= \alpha_0^2 + \sum_{k=1}^{\infty}(k+1)\alpha_{-k} \cdot \alpha_k + \sum_{k=1}^{\infty} \alpha_{-1-(k-1)} \cdot (-(k-1))\alpha_{(k-1)+1} \\
&= \alpha_0^2 + 2\sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k
\end{aligned}$$