$$
\begin{aligned}
[L_{-2}, L_2] &= \sum_{n=2}^{\infty}([L_{-2}, \alpha_{2-n}] \cdot \alpha_n + \alpha_{2-n} \cdot [L_{-2}, \alpha_n]) + \frac{1}{2}([L_{-2}, \alpha_1] \cdot \alpha_1 + \alpha_1 \cdot [L_{-2}, \alpha_1]) \\
&= \sum_{n=2}^{\infty}((n-2)\alpha_{-n} \cdot \alpha_n + \alpha_{2-n} \cdot (-n)\alpha_{n-2}) + \frac{1}{2}((-1)\alpha_{-1} \cdot \alpha_1 + \alpha_1 \cdot (-1)\alpha_{-1}) \\
&= \sum_{n=2}^{\infty}(n-2)\alpha_{-n} \cdot \alpha_n + \sum_{n=0}^{\infty}\alpha_{-n} \cdot (-n-2)\alpha_n \\
&\quad -\frac{1}{2}(\alpha_{-1} \cdot \alpha_1 + \alpha_1 \cdot \alpha_{-1}) \\
&= -4\sum_{n=2}^{\infty}\alpha_{-n} \cdot \alpha_n + \alpha_{-0} \cdot (-0-2)\alpha_0 + \alpha_{-1} \cdot (-1-2)\alpha_1 \\
&\quad -\frac{1}{2}(2\alpha_{-1} \cdot \alpha_1 + [\alpha_1, \alpha_{-1}]) \\
&= -4\sum_{n=1}^{\infty}\alpha_{-n} \cdot \alpha_n - 2\alpha_0^2 - \frac{1}{2}g_\mu^\mu \\
&= -4L_0 - \frac{g_\mu^\mu}{2}
\end{aligned}
$$