

$$\begin{aligned}
[L_{m+1}, L_{-m-1}] &= \frac{1}{m-1} [[L_m, L_1], L_{-m-1}] \\
&= \frac{-1}{m-1} ([[L_1, L_{-m-1}], L_m] + [[L_{-m-1}, L_m], L_1]) \\
&= \frac{-1}{m-1} ((1+m+1)L_{-m}, L_m] + [(-m-1-m)L_{-1}, L_1]) \\
&= \frac{m+2}{m-1} [L_m, L_{-m}] - \frac{2m+1}{m-1} [L_1, L_{-1}] \\
&= \frac{m+2}{m-1} (2mL_0 + \frac{c}{12}(m-1)m(m+1)) - \frac{2m+1}{m-1} 2L_0 \\
&= 2(m+1)L_0 + \frac{c}{12}m(m+1)(m+2)
\end{aligned}$$